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C^*-ALGEBRA FIBRE BUNDLES[1]

BY

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ABSTRACT. It will be shown in this paper that for any C^*-algebra fibre bundle with base space X and fibre A, a C^*-algebra, the Jacobson spectrum of the C^*-algebra of sections of the fibre bundle can be identified as a topological fibre bundle with the same base space X and fibre the Jacobson spectrum of A.

A C^*-algebra fibre bundle $\Sigma = (E, X, A, p, U, \phi_u, G)$ is specified up to equivalence [3] by a topological bundle space E, a locally compact Hausdorff base space X, a C^*-algebra fibre A, a continuous projection $p\colon E \to X$, an open covering U of X, and homeomorphisms $\phi_u\colon u \times A \to p^{-1}(u)$ for $u \in U$. The ϕ_u are fibre-preserving: $\phi_u(x, A) = p^{-1}(x)$. Furthermore, there is an effective topological group G of *-automorphisms of the fibre A. The mappings ϕ_u and the fibre A are related as follows:

For $x \in u \cap v$, $u,v \in U$, and $a \in A$, let

$$\phi_u^{-1}\phi_v(x, a) = (x, g_{uv}(x)(a)).$$

Then $g_{uv}(x) \in G$, and the map $g_{uv}\colon u \cap v \to G$ is continuous. If $y \in p^{-1}(x)$, $x \in u \in U$, the relation is described by writing $\phi_u^{-1}(y) = (x, t_u(y))$. Let S be the set of continuous sections $\gamma\colon X \to E$ $(p\gamma(x) = x)$ and $D = \{\gamma \in S\colon |\gamma(x)| = \|t_u(\gamma(x))\|_A$ vanishes at infinity$\}$. Note that $|\gamma(x)| = \|t_u(\gamma(x))\|_A$ is independent of the choice of u in U containing x, since $g_{uv}(x) \in G$ is an isometry.

For $\gamma_1,\gamma_2 \in D$, if $x \in u \in U$, define

$$(\gamma_1 + \gamma_2)(x) = \phi_u(x, t_u(\gamma_1(x)) + t_u(\gamma_2(x))),$$

and

$$(\gamma_1\gamma_2)(x) = \phi_u(x, t_u(\gamma_1(x))t_u(\gamma_2(x))).$$

If $x \in v \in U$ also, then

$$\begin{aligned}(\gamma_1 + \gamma_2)(x) &= \phi_v(x, t_v(\gamma_1(x)) + t_v(\gamma_2(x))) \\ &= \phi_v(x, g_{vu}(x)(t_u(\gamma_1(x)) + t_u(\gamma_2(x)))) \\ &= \phi_u(x, t_u(\gamma_1(x)) + t_u(\gamma_2(x))).\end{aligned}$$

Thus $(\gamma_1 + \gamma_2)(x)$ is well defined as is $(\gamma_1\gamma_2)(x)$; clearly, $\gamma_1 + \gamma_2$ and $\gamma_1\gamma_2$ belong to D.

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For $\gamma \in D$, since $|\gamma(x)|$ is continuous on X,

$$\|\gamma\| \equiv \sup_{x \in X} |\gamma(x)| < \infty.$$

Direct verification shows that $\|\cdot\|$ is a norm on D and that with respect to this norm and the operations as defined above, and the involution $\gamma^*(x) \equiv \phi_u(x, (t_u(\gamma(x)))^*)$, D is a C^*-algebra, called the C^*-algebra of sections of the fibre bundle Σ.

LEMMA 1. *Let $\Sigma = (E, X, A, p, U, \phi_u, G)$ be a C^*-algebra fibre bundle and let D be its C^*-algebra of sections. If $u \in U$, and $f \in C_{00}(X, A)$ with* support$(f) \subset u$, *define*

$$\gamma(x) = \phi_u(x, f(x)), \quad \text{if } x \in u,$$
$$= O(x), \quad \text{if } x \notin u,$$

where $O(x) = \phi_v(x, 0)$, if $x \in v \in U$. Then $\gamma \in D$ and $t_u(\gamma(x)) = f(x)$, whence $A = \{t_u(\gamma(x)): \gamma \in D\}$ for each $x \in X$, $u \in U$ with $x \in u$.

PROOF. Cf. [**2**, Lemma 1.1].

LEMMA 2. *Let $\Sigma = (E, X, A, p, U, \phi_u, G)$ be a C^*-algebra fibre bundle and let D be its C^*-algebra of sections. If I is a closed two-sided ideal of D, for each $x \in X$ and $u \in U$ such that $x \in u$, set $I_u(x) = \{t_u(\gamma(x)): \gamma \in I\}$. Then $I = \{\gamma \in D$: for every $x \in X$, there exists in U a u with $x \in u$ and such that $t_u(\gamma(x)) \in I_u(x)\}$.*

PROOF. If $\gamma \in I$, for every $x \in X$ and $u \in U$ with $x \in u$, then $t_u(\gamma(x)) \in I_u(x)$ by definition of $I_u(x)$.

Conversely, if $\gamma \in D$ and if for every $x \in X$, there exists a u in U with $x \in u$ such that $t_u(\gamma(x)) \in I_u(x)$. If $\varepsilon > 0$, $K \equiv \{x \in X: |\gamma(x)| \geq \varepsilon\}$ is compact. For every y in K, there exists a v in U with $y \in v$ and there exists a z_y in I such that $t_v(\gamma(y)) = t_v(z_y(y))$. Since A has an approximate identity, there exists an a_y in A such that

$$\|t_v(z_y(y)) - a_y(t_v(z_y(y)))\| \leq \varepsilon.$$

Let $f \in C_{00}(X)$ be such that $f(y) = 1$ and support $(f) \subset v$. Define

$$w_y(x) = \phi_v(x, f(x)a_y), \quad \text{if } x \in v,$$
$$= O(x), \quad \text{if } x \notin v.$$

Then $w_y \in D$ and $t_v(w_y(y)) = f(y)a_y = a_y$ (Lemma 1), and

$$|\gamma(y) - w_y z_y(y)| = \|t_v(\gamma(y)) - t_v(w_y(y))t_v(z_y(y))\|$$
$$= \|t_v(z_y(y)) - a_y t_y(z_y(y))\| < \varepsilon.$$

By the continuity of $x \to \gamma(x)$ on X, $|\gamma(x) - (w_y z_y)(x)| < \varepsilon$ on a neighborhood of y. Therefore, there is a finite open covering $O_1, O_2, \ldots, O_n$ of K such that $|\gamma(x) - (w_i z_i)(x)| < \varepsilon$ on O_i for $i = 1,2,\ldots,n$, and there are $g_1, g_2, \ldots, g_n$ constituting a partition of unity subordinate to $O_1, O_2, \ldots, O_n$, i.e., $g_i \in C_{00}(X)$, g_i vanishes outside O_i and $g_1 + g_2 + \cdots + g_n = 1$ on K. Then $\sum_{i=1}^n g_i w_i z_i \in I$, since $g_i w_i \in D$ and $z_i \in I$, and

$$\left|\gamma(x) - \sum_{i=1}^{n} (g_i w_i z_i)(w)\right| < \varepsilon \quad \text{on } K,$$

thus $\|\gamma - \sum_{i=1}^{n} g_i w_i z_i\| < \varepsilon$. Whence $\gamma \in I$, and the proof is complete.

Let A be a C^*-algebra. We denote by $\hat{A}$ the set of equivalence classes of irreducible representations of A endowded with the Jacobson topology [**1**, 3.1.1].

THEOREM 3. *Let $\Sigma = (E, X, A, p, U, \phi_u, G)$ be a C^*-algebra fibre bundle and let D be its C^*-algebra of sections. Then to each $(x, u, \pi) \in X \times U \times \hat{A}$ with $x \in u$, there corresponds a σ in $\hat{D}$ such that $\sigma(\gamma) = \pi(t_u(\gamma(x)))$, for $\gamma \in D$. Conversely, every σ in $\hat{D}$ is of this form.*

PROOF. If $(x, u, \pi) \in X \times U \times \hat{A}$ with $x \in u$, then since $A = \{t_u(\gamma(x)) \colon \gamma \in D\}$ (Lemma 1), $\sigma(\gamma) \equiv \pi(t_u(\gamma(x)))$, $\gamma \in D$, is an irreducible representation of D. Conversely, if $\sigma \in \hat{D}$, put $I = \ker \sigma$ and set $S = \{x \in X \colon I_u(x) \neq A$, for every $u \in U$ with $x \in u\}$. If $S = \varnothing$, then for every x in X, there exists in U a u with $x \in u$ such that $I_u(x) = A$. It follows from Lemma 2 that $I = A$, a contradiction, hence $S \neq \varnothing$.

Suppose x and y are distinct elements in S. Let u and v be in U such that $x \in u$ and $y \in v$. Choose two open subsets u_1 and v_1 in X such that $u_1 \cap v_1 = \varnothing$ and $x \in u_1 \subset u$, $y \in v_1 \subset v$. Define $M = \{\gamma \in D \colon \gamma(z) = \phi_u(z, f(z))$, if $z \in u$, $= O(z)$, if $z \notin u$, where $f \in C_{00}(X, A)$ with support $(f) \subset u_1\}$. Similarly, let $N = \{\gamma \in D \colon \gamma(z) = \phi_v(z, g(z))$, if $z \in v$, $= O(z)$, if $z \notin v$, where $g \in C_{00}(X, A)$ with support $(g) \subset v_1\}$. Then M and N are closed two-sided ideals of D, and if $\gamma_1\gamma_2 \in M \cdot N$,

$$\begin{aligned} \|\gamma_1\gamma_2\| &= \sup_{x \in X} |(\gamma_1\gamma_2)(x)| = \sup_{x \in X} \|t_u(\gamma_1(x)) t_u(\gamma_2(x))\| \\ &\leqslant \sup_{x \in X} \|t_u(\gamma_1(x))\| \cdot \|t_u(\gamma_2(x))\| = 0, \end{aligned}$$

hence $0 = M \cdot N \subset I$. Since $I = \ker \sigma$ is a primitive ideal of D, by [**1**, 2.11.4], $M \subset I$ or $N \subset I$. Suppose $M \subset I$, then $M_u(x) \subset I_u(x)$. Note that $A = M_u(x)$, since if $a \in A$, by the construction of the ideal M there exists a γ in M such that $a = t_u(\gamma(x))$, and hence $A = I_u(x)$ which contradicts x in S. Therefore $S = \{x_0\}$ is a one-point set. Owing to Lemma 1, it follows that σ induces a π in $\hat{A}$ according to

$$\pi(t_u(\gamma(x_0))) = \sigma(\gamma), \quad \text{for } \gamma \in D.$$

This shows that for every σ in $\hat{D}$ there corresponds (x, u, π) in $X \times U \times \hat{A}$ such that $\sigma(\gamma) = \pi(t_u(\gamma(x_0)))$ for $\gamma \in D$.

The correspondence between $X \times U \times \hat{A}$ and $\hat{D}$ in Theorem 3 is not necessarily bijective, although to each σ in $\hat{D}$ there corresponds a unique x in X. It will be shown that in the Jacobson topology on $\hat{D}$, $\hat{D}$ is a topological fibre bundle over X with fibre $\hat{A}$ and covering U.

LEMMA 4. *Let $\Sigma = (E, X, A, p, U, \phi_u, G)$ be a C^*-algebra fibre bundle and let D be its C^*-algebra of sections. For σ in $\hat{D}$ there exists according to Theorem 3 a unique x in X such that σ corresponds to (x, u, π). Define p on $\hat{D}$ into X by $\tilde{p}(\sigma) = x$, then $\tilde{p}$ is a continuous projection.*

PROOF. It is immediate that $\tilde{p}$ so defined is surjective.

Assume the net $\{\sigma_\lambda\}$ converges to σ in $\hat{D}$. Choose $(x_\lambda, u_\lambda, \pi_\lambda)$ and (x, u, π) in $X \times U \times \hat{A}$ such that

$$\sigma_\lambda(\gamma) = \pi_\lambda\big(t_{u_\lambda}(\gamma(x_\lambda))\big) \quad \text{and} \quad \sigma(\gamma) = \pi\big(t_u(\gamma(x))\big), \qquad \gamma \in D.$$

Note that x_λ and x are uniquely determined.

If $x_\lambda \not\to x$ in X, there exist a subnet $\{x_{\lambda'}\}$ and a function f in $C_{00}(X)$ with support $(f) \subset u$ such that $f(x_{\lambda'}) = 0$ and $f(x) = 1$. Furthermore, pick an a in A such that $\pi(a) \neq 0$. Define

$$\begin{aligned}\gamma(y) &= \phi_u(y, f(y)a), \quad \text{if } y \in u,\\ &= O(y), \quad \text{if } y \neq u,\end{aligned}$$

then $\gamma \in D$ and $t_u(\gamma(x)) = f(x)a = a$,

$$\begin{aligned}t_u\big(\gamma(x_{\lambda'})\big) &= f(x_{\lambda'})a = 0, \quad \text{if } x_{\lambda'} \in u,\\ &= 0, \quad \text{if } x_{\lambda'} \notin u,\end{aligned}$$

hence $t_u(\gamma(x_{\lambda'})) = 0$. Moreover,

$$\begin{aligned}t_{u_{\lambda'}}\big(\gamma(x_{\lambda'})\big) &= g_{u_{\lambda'}u}(x_{\lambda'})\big(t_u(\gamma(x_{\lambda'}))\big) = g_{u_{\lambda'}u}(x_{\lambda'})(0) = 0, \quad \text{if } x_{\lambda'} \in u,\\ &= 0, \quad \text{if } x_{\lambda'} \notin u.\end{aligned}$$

Thus

$$t_{u_{\lambda'}}\big(\gamma(x_{\lambda'})\big) = 0 \quad \text{and} \quad \pi_{\lambda'}\big(t_{u_{\lambda'}}(\gamma(x_{\lambda'}))\big) = 0,$$

therefore, $\gamma \in \bigcap_{\lambda'} \ker \sigma_{\lambda'}$. Since $\sigma_{\lambda'} \to \sigma$ in $\hat{D}$, it follows that $\bigcap_{\lambda'} \ker \sigma_{\lambda'} \subset \ker \sigma$, by [**1**, 3.4.4 and 3.4.10]. Whence $\gamma \in \ker \sigma$, but $\sigma(\gamma) = \pi(t_u(\gamma(x))) = \pi(a) \neq 0$, a contradiction. Therefore $\tilde{p}$ is continuous.

For each u in U, define $\tilde{\phi}_u$: $u \times \hat{A} \to \tilde{p}^{-1}(u)$ by $\tilde{\phi}_u(x, \pi)(\gamma) = \pi(t_u(\gamma(x)))$, $\gamma \in D$.

LEMMA 5. *Use the above notations. For every u in U, $\tilde{\phi}_u$ is a homeomorphism from $u \times \hat{A}$ onto $\tilde{p}^{-1}(u)$ and is fibre-preserving.*

PROOF. If $\sigma \in \tilde{p}^{-1}(u)$, there exists (Theorem 3) a unique x in X and some v in U with $x \in v \cap u$ such that $\sigma(\gamma) = \pi(t_v(\gamma(x)))$, $\gamma \in D$. Let δ be defined on A by $\delta(a) = \pi(g_{vu}(x)(a))$, $a \in A$. Then $\delta \in \hat{A}$ and

$$\delta\big(t_u(\gamma(x))\big) = \pi\big(g_{vu}(x)\big(t_u(\gamma(x))\big)\big) = \pi\big(t_v(\gamma(x))\big) = \sigma(\gamma), \quad \text{for } \gamma \in D.$$

Thus $\tilde{\phi}_u(x, \delta) = \sigma$, whence $\tilde{\phi}_u$ is surjective.

To prove $\tilde{\phi}_u$ is 1-1, assume $(x, \pi) \neq (y, \sigma)$ in $u \times \hat{A}$. If $x \neq y$, there exists a f in $C_{00}(X)$ with support $(f) \subset u$ such that $f(x) = 1$ and $f(y) = 0$. Choose an a in A such that $\pi(a) \neq 0$. Define

$$\begin{aligned}\gamma(z) &= \phi_u(z, f(z)a), \quad \text{if } z \in u,\\ &= O(z), \quad \text{if } z \notin u.\end{aligned}$$

Then

$$\pi(t_u(\gamma(x))) = \pi(f(x)a) = \pi(a) \neq 0,$$
$$\sigma(t_u(\gamma(y))) = \sigma(f(y)a) = \sigma(0) = 0.$$

Hence $\tilde{\phi}_u(x, \pi) \neq \tilde{\phi}_u(y, \sigma)$.

If $x = y$, then $\pi \neq \sigma$ in $\tilde{p}^{-1}(u)$. Hence $\tilde{\phi}_u(x, \pi) \neq \tilde{\phi}_u(y, \sigma)$. This proves that $\tilde{\phi}_u$ is biijective.

Suppose the net $(x_\lambda, \pi_\lambda) \to (x, \pi)$ in $u \times \hat{A}$. If $\gamma \in \bigcap_\lambda \ker \tilde{\phi}_u(x_\lambda, \pi_\lambda)$, $\pi_\lambda(t_u(\gamma(x_\lambda))) = 0$. Since $\phi_u^{-1}(y) = (x, t_u(y))$, $y \to t_u(y)$ is continuous and since $x_\lambda \to x$ in X and γ is continuous, $\gamma(x_\lambda) \to \gamma(x)$ in E. Hence, if $\varepsilon > 0$, there exists an λ_0 such that if $\lambda > \lambda_0$, $\|t_u(\gamma(x_\lambda)) - t_u(\gamma(x))\| < \varepsilon$,

$$\|\pi_\lambda(t_u(\gamma(x)))\| = \|\pi_\lambda(t_u(\gamma(x_\lambda))) - \pi_\lambda(t_u(\gamma(x)))\|$$
$$\leqslant \|t_u(\gamma(x_\lambda)) - t_u(\gamma(x))\| < \varepsilon.$$

Therefore, $\|t_u(\gamma(x))/\bigcap_{\lambda>\lambda_0} \ker \pi_\lambda\| = \sup_{\lambda>\lambda_0} \|t_u(\gamma(x))/\ker \pi_\lambda\| < \varepsilon$. Since $\pi_\lambda \to \pi$ in $\hat{A}$, hence $\bigcap_{\lambda>\lambda_0} \ker \pi_\lambda \subset \ker \pi$ [**1**, 3.4.4 and 3.4.10]. Thus $\|t_u(\gamma(x))/\ker \pi\| < \varepsilon$. Since ε is arbitrary, $t_u(\gamma(x)) \in \ker \pi$, $\tilde{\phi}_u(x, \pi)(\gamma) = \pi(t_u(\gamma(x))) = 0$, whence $\gamma \in \ker \tilde{\phi}_u(x, \pi)$. This shows that $\bigcap_\lambda \ker \tilde{\phi}_u(x_\lambda, \pi_\lambda) \subset \ker \tilde{\phi}_u(x, \pi)$. Hence $\tilde{\phi}_u(x, \pi) \in \overline{\{\tilde{\phi}_u(x_\lambda, \pi_\lambda)\}}$ in $\hat{D}$, and the same holds for every subnet of (x_λ, π_λ). Hence $\tilde{\phi}_u(x_\lambda, \pi_\lambda) \to \tilde{\phi}_u(x, \pi)$ in $\tilde{p}^{-1}(u)$.

Conversely, assume $\tilde{\phi}_u(x_\lambda, \pi_\lambda) \to \tilde{\phi}_u(x, \pi)$ in $\tilde{p}^{-1}(u)$. If $x_\lambda \not\to x$ in X, there exist a subnet $\{x_{\lambda'}\}$ and a function f in $C_{00}(X)$ with support$(f) \subset u$ such that $f(x_{\lambda'}) = 0$ and $f(x) = 1$. Let a in A be such that $\pi(a) \neq 0$. Define

$$\gamma(y) = \phi_u(y, f(y)a), \quad \text{if } y \in u,$$
$$= O(y), \quad \text{if } y \notin u,$$

then $\gamma \in D$ and

$$\pi_{\lambda'}(t_u(\gamma(x_{\lambda'}))) = \pi_{\lambda'}(f(x_{\lambda'})a) = \pi_{\lambda'}(0) = 0,$$
$$\pi(t_u(\gamma(x))) = \pi(f(x)a) = \pi(a) \neq 0.$$

Hence $\gamma \in \bigcap_{\lambda'} \ker \tilde{\phi}_u(x_{\lambda'}, \pi_{\lambda'})$, but $\gamma \notin \ker \tilde{\phi}_u(x, \pi)$, which contradicts $\tilde{\phi}_u(x_\lambda, \pi_\lambda) \to \tilde{\phi}_u(x, \pi)$. Thus $x_\lambda \to x$ in X.

Next, it will be shown $\pi_\lambda \to \pi$ in $\hat{A}$. If $a \in \bigcap_\lambda \ker \pi_\lambda$, $\pi_\lambda(a) = 0$ for all λ. By Lemma 1, there exists a γ in D such that $a = t_u(\gamma(x))$. It has been shown that $x_\lambda \to x$ in X, so that for $\varepsilon > 0$ there exists an λ_0 such that if $\lambda > \lambda_0$ then

$$\|t_u(\gamma(x_\lambda)) - t_u(\gamma(x))\| < \varepsilon,$$

and so

$$\|\pi_\lambda(t_u(\gamma(x_\lambda)))\| = \|\pi_\lambda(t_u(\gamma(x_\lambda))) - \pi_\lambda(t_u(\gamma(x)))\|$$
$$\leqslant \|t_u(\gamma(x_\lambda)) - t_u(\gamma(x))\| < \varepsilon,$$

whence

$$\|\gamma/\ker \tilde{\phi}_u(x_\lambda, \pi_\lambda)\| = \|\pi_\lambda(t_u(\gamma(x_\lambda)))\| < \varepsilon, \quad \text{for } \lambda > \lambda_0.$$

Thus $\|\gamma/\bigcap_{\lambda>\lambda_0} \ker \tilde{\phi}_u(x_\lambda, \pi_\lambda)\| < \varepsilon$.

Since $\tilde{\phi}_u(x_\lambda, \pi_\lambda) \to \tilde{\phi}_u(x, \pi)$, $\bigcap_{\lambda>\lambda_0} \ker \tilde{\phi}_u(x_\lambda, \pi_\lambda) \subset \ker \tilde{\phi}_u(x, \pi)$. Hence $\|\gamma/\tilde{\phi}_u(x, \pi)\| < \varepsilon$ and therefore $\gamma \in \tilde{\phi}_u(x, \pi)$, i.e., $\pi(t_u(\gamma(x))) = \pi(a) = 0$. Thus $a \in \ker \pi$, whence $\bigcap_\lambda \ker \pi_\lambda \subset \ker \pi$. It follows that $\pi \in \overline{\{\pi_\lambda\}}$ in $\hat{A}$ by [1, 3.4.4 and 3.4.10], and the same holds for every subnet of $\{\pi_\lambda\}$. Therefore $\pi_\lambda \to \pi$ in $\hat{A}$, and this proves that $\tilde{\phi}_u$ is bicontinuous.

Finally, if $x \in u$ and σ is in $\tilde{p}^{-1}(x)$, σ corresponds by Theorem 3 to some (x, v, π) in $X \times U \times \hat{A}$ such that

$$\sigma(\gamma) = \pi(t_v(\gamma(x))), \quad \text{for } \gamma \in D.$$

Since $\pi \in \hat{A}$, $\pi g_{vu}(x) \in \hat{A}$, and

$$\begin{aligned}(\tilde{\phi}_u(x, \pi g_{vu}(x)))(\gamma) &= (\pi g_{vu}(x))(t_u(\gamma(x))) \\ &= \pi(g_{vu}(x)(t_u(\gamma(x)))) = \pi(t_v(\gamma(x))) = \sigma(\gamma).\end{aligned}$$

Hence $\tilde{\phi}_u(x, \hat{A}) = \tilde{p}^{-1}(x)$ and thus $\tilde{\phi}_u$ is fibre-preserving.

If $g \in G$, and if the map $g^*\colon \hat{A} \to \hat{A}$ is defined by $g^*(\pi)(a) = \pi(g(a))$, then g^* is bijective. Furthermore, if the net $\pi_\lambda \to \pi$ in $\hat{A}$, and if $a \in \bigcap_\lambda \ker g^*(\pi_\lambda)$, $\pi_\lambda(g(a)) = 0$. Hence $g(a) \in \bigcap_\lambda \ker \pi_\lambda$. Since $\pi_\lambda \to \pi$ in $\hat{A}$, $\bigcap_\lambda \ker \pi_\lambda \subset \ker \pi$, by [1, 3.4.4 and 3.4.10]. Whence $g(a) \in \ker \pi$, $a \in \ker g^*(\pi)$. Thus $\bigcap_\lambda \ker g^*(\pi_\lambda) \subset \ker g^*(\pi)$, $g^*(\pi) \in \overline{\{g^*(\pi_\lambda)\}}$ in $\hat{A}$, and the same holds for every subnet of $\{g^*(\pi_\lambda)\}$. Therefore $g^*(\pi_\lambda) \to g^*(\pi)$ and thus g^* is continuous. A similar argument shows $(g^*)^{-1}$ is continuous. Thus g^* belongs to Auteo$(\hat{A})$, the group of self-homeomorphisms of $\hat{A}$.

Consider the group $\tilde{G}$ of self-homeomorphisms of the form g^* where $g \in G$. The map $T\colon G \to \tilde{G}$ defined by $T(g) = g^*$ is a group antiepimorphism $(T(gh) = T(h)T(g))$. If the quotient group $G/\ker T$ is given the quotient topology, $G/\ker T$ is a topological group. Since

$$g/\ker T \to g^*$$

is an anti-isomorphism from $G/\ker T$ onto $\tilde{G}$, we topologize $\tilde{G}$ by giving it the topology derived from $G/\ker T$, i.e., a set $S^* \subset \tilde{G}$ is open if and only if the preimage S is open in $G/\ker T$. Then $\tilde{G}$ becomes a topological group.

If $u, v \in U$ and $x \in u \cap v$, define

$$\tilde{g}_{uv}(x)\colon \hat{A} \to \hat{A}$$

by

$$\tilde{g}_{uv}(x)(\pi)(a) = \pi(g_{vu}(x)(a)), \quad \text{for } \pi \in \hat{A}, a \in A.$$

Then $\tilde{g}_{uv}(x) = (g_{vu}(x))^* \in \tilde{G}$.

Since the map $g_{uv}\colon u \cap v \to G$ is continuous, and the canonical map: $G \to G/\ker T$ is also continuous, it follows that the map $\tilde{g}_{uv}\colon u \cap v \to \tilde{G}$ is continuous. Moreover,

$$\begin{aligned}\tilde{\phi}_v(x, \pi)(\gamma) &= \pi(t_v(\gamma(x))) = \pi(g_{vu}(x)(t_u(\gamma(x)))) \\ &= \tilde{g}_{uv}(x)(\pi)(t_u(\gamma(x))) \\ &= \tilde{\phi}_u(x, \tilde{g}_{uv}(x)(\pi))(\gamma), \quad \text{for } \gamma \in D.\end{aligned}$$

Thus $\tilde{\phi}_u^{-1}\tilde{\phi}_v(x, \pi) = (x, \tilde{g}_{uv}(x)(\pi))$.

Owing to Lemmas 4 and 5, the following result may be demonstrated:

THEOREM 6. *Let* $\Sigma = (E, X, A, p, U, \phi_u, G)$ *be a* C^**-algebra fibre bundle over a locally compact Hausdorff base space* X*, with fibre* A*, a* C^**-algebra. Then* Σ *defines a topological fibre bundle* $\hat{\Sigma} = (\hat{D}, X, \hat{A}, \tilde{p}, U, \tilde{\phi}_u, \tilde{G})$ *over the same base space* X*, with fibre the Jacobson spectrum* $\hat{A}$ *of* A *and bundle the Jacobson spectrum* $\hat{D}$ *of the* C^**-algebra of sections of* Σ.

PROOF. By the above argument, the following results are readily derived:

(i) $\tilde{p}$ is a continuous projection from $\hat{D}$ onto X,

(ii) $\tilde{\phi}_u$ is a homeomorphism from $u \times \hat{A}$ onto $\tilde{p}^{-1}(u)$ and is fibre-preserving in that $\tilde{\phi}_u(x, \hat{A}) = \tilde{p}^{-1}(x)$,

(iii) $\tilde{\phi}_u^{-1}\tilde{\phi}_v(x, \pi) = (x, \tilde{g}_{uv}(x)(\pi))$ and $\tilde{g}_{uv}(x) \in \tilde{G}$, and the map $\tilde{g}_{uv}$: $u \cap v \to \tilde{G}$ is continuous.

Therefore $\hat{\Sigma} = (\hat{D}, X, \hat{A}, \tilde{p}, U, \tilde{\phi}_u, \tilde{G})$ is a topological fibre bundle.

REMARK. It is shown in [2] that if Σ is a Banach algebra fibre bundle over a compact Hausdorff base space X with fibre A a so-called Q-uniform Banach algebra then the maximal ideal space (corresponding to $\hat{D}$) of the Banach algebra of sections of Σ with suitable topology can be identified as a topological fibre bundle with base space X and fibre the set of maximal ideals of the Banach algebra A. An example is given there to show that if the Jacobson topologies are used for the maximal ideal space of the Banach algebra of sections of Σ and the maximal ideal space of A, the coordinate functions $\tilde{\phi}_u$ (as in Lemma 5) need not be continuous.

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